Homework 8, due 11/19

1. Let $f : X \to Y$ be a non-constant holomorphic map between Riemann surfaces. Let $p \in X$, and $q = f(p) \in Y$. Show that there exist local coordinates z, w centered at p, q respectively, such that in these coordinates the map f is given by $f(z) = z^k$ for an integer $k > 0$.

2. Suppose that z, w are two local coordinates on a Riemann surface, such that $z = 2w + 2w^2$ in a neighborhood of the origin. Let $\alpha = (z^{-2} + z^{-1})dz$ be a meromorphic 1-form. Show that in terms of the coordinate w we have
$$\alpha = \left(\frac{1}{2}w^{-2} + w^{-1} + \frac{1}{2} + O(w)\right) dw.$$

3. Show that the open mapping theorem holds for holomorphic maps between Riemann surfaces. I.e. if $f : X \to Y$ is a non-constant holomorphic map, then $f(U) \subset Y$ is open for all open $U \subset X$.

4. Let $f : X \to Y$ be a non-constant holomorphic map between Riemann surfaces such that X is compact. Show that then Y is compact and f is surjective.